Fatigue of Kidney Stones with Heterogeneous Microstructure Subjected to Shock-Wave Lithotripsy

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Abstract: In this article a theoretical framework is developed for the mechanics of kidney stones with an isotropic, random microstructure—such as those comprised of cystine or struvite. The approach is based on a micromechanical description of kidney stones comprised of crystals in a binding matrix. Stress concentration functions are developed to determine load sharing of the particle phase and the binding matrix phase. Measurements have shown the inclusions to be considerably harder than the binder; consequently, loading of a stone leads to higher stresses in the inclusions than in the binder. As an illustration of the theory, the fatigue of kidney stones subject to shock-wave lithotripsy is considered. Stress concentration functions are used to construct fatigue-life estimates for each phase, as a function of the volume fraction and of the mechanical properties of the constituents, as well as the loading from SWL. The failure of the binding matrix, or of the particulate phase, is determined explicitly in a model for the accumulation of distributed damage. The theory can be used to assess the importance of microscale heterogeneity on the comminution of renal calculi, and to estimate the number of cycles to failure in terms of measurable material properties.

INTRODUCTION

Shock-wave lithotripsy (SWL) has become the primary technique to treat kidney stones. A large percentage of kidney stones are treated by SWL alone and SWL combined with other treatments. Thousands of shocks are required to break stones to small enough sizes for natural elimination from the body.

There remains much scientific investigation on the precise mechanisms of stone comminution. However, central to the process appears to be the passage of stress waves through the stone, and the erosion of stone surface by cavitation damage from imploding microbubbles.\(^1\)\(^2\) It may be that a number of mechanisms interact to produce breakup, working in concert with one another. Whatever the origin of the stresses, the process of stone comminution has been described as one of brittle fracture, for example in Lokhandwalla and Sturtevant\(^3\) and in Coleman and Saunders.\(^4\) As such, the process has been analyzed with the use of a one-dimensional cohesive zone model for a homogeneous material that can fragment into two pieces.\(^3\) Lokhandwalla and Sturtevant were able to estimate the number of shock waves to failure.

The present work is similar in spirit to that of Lokhandwalla and Sturtevant, with a number of further enhancements. Here a theoretical framework capable of treating the heterogeneous microstructure of real kidney stones is presented. A three-dimensional material that is heterogeneous on the microscale is considered. The approach is based on micromechanical description of those types of kidney stones (reviewed below) comprised of agglomerated isotropic particles, with a binding matrix (Figure 1). The presence of a binder phase and inclusions leads to stress concentrations (or differential load sharing) in a mean sense that can be estimated. After the stress concentrations for material properties derived from experiments are developed, fatigue damage of an initially flawless sample of material is considered. The goal is to analyze the development of damage in an overall sense. Stress concentration functions are used to construct fatigue-life estimates for each phase, as a function of the volume fraction and of the mechanical properties of the constituents, as well as the mean and fluctuating loading. The failure of the binding matrix, which is the key requirement to break up the stone, can be predicted explicitly. The main result is to determine the influence of the stress concentrations on the ultimate resistance to breakup of the stone. Hopefully this will begin a theoretical discussion of the relative ease of breaking different types of stones as a function of the microstructure. This has been a subject of experimental inquiry by a number of authors,\(^5\)\(^6\) see also the review by Coleman and Saunders.\(^4\)
One difficulty is that in order to apply well-developed techniques, some unknown parameters are needed. A secondary purpose of the present work is to describe what kidney-stone measurements are needed in order to have a solid foundation for a sound theory. However, this will require improvisation in order to develop a concrete result. What has been measured is the sound speed through the crystalline material and the matrix of uric-acid stones; this was accomplished by acoustical measurements of $p$-wave speeds by Pittomvils et al.\textsuperscript{13}—see also Reference 6. These data can be used to determine the bulk modulus of each phase. From the $p$-wave speed $(c)$,

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}} \Rightarrow \lambda + 2\mu = c^2\rho,$$ \hspace{1cm} (1)

where $\lambda$ and $\mu$ are the Lamé parameters and $\rho$ is the mass density. Thus, because $\lambda = \kappa - 2/3\mu$,

$$\kappa + \frac{4}{3\mu} = c^2\rho,$$ \hspace{1cm} (2)

where $\kappa$ is the bulk modulus. The shear moduli can be estimated by assuming that the material has a shear property similar to that of a ceramic, due to the polycrystalline nature of the materials involved. In other words, the ratio of bulk to shear moduli is a function of Poisson’s ratio $\nu$ taken to be a representative value of $\nu = 0.35$. The relationship between $\kappa$ and $\mu$ is

$$\frac{\kappa}{\mu} = \frac{2(1 + \nu)}{3(1 - 2\nu)};$$ \hspace{1cm} (3)

thus $\kappa = 3\mu$. Consequently,

$$\kappa = \frac{1}{2}c^2\rho \quad \text{and} \quad \mu = \frac{1}{6}c^2\rho.$$ \hspace{1cm} (4)

The values of $c$ and $\rho$ for each phase were given in Pittomvils et al.\textsuperscript{13} With the parameters determined as indicated, the estimated moduli are shown in Table I. It is unfortunate that data for uric-acid stones must be used,\textsuperscript{13} and yet the theory developed is more suitable for stones with an isotropic microstructure. However, that appears to be all that is available.

**Notation**

Before embarking on the theory, a word about notation is in order. Throughout this work, boldface symbols indicate vectors or tensors. The inner product of two vectors $(\mathbf{u})$ and $(\mathbf{v})$ is denoted $\mathbf{u} \cdot \mathbf{v}$. At the risk of oversimplification, the distinction between second-order tensors and matrices is ignored. Furthermore, a Cartesian basis exclusively employed. Readers may consult the texts of Malvern\textsuperscript{14} and Marsden and Hughes\textsuperscript{15} for background. Hence, if the second-order tensor $\mathbf{A}$ is considered, with its matrix representation

**MICROMECHANICAL CHARACTERIZATION OF A HETEROGENEOUS STONE**

*Structure and Morphology of Heterogeneous Stones*

Kidney stones consist of an agglomeration of crystalline materials interspersed with an organic matrix of fibrous quality.\textsuperscript{4,5,7} The crystalline phase is usually calcium oxalate monohydrate (COM) (CaC$_2$O$_4$H$_2$O), calcium oxalate dihydrate (COD), (CaC$_2$O$_4$2H$_2$O), uric acid (C$_5$H$_4$N$_4$O$_3$), struvite (MgNH$_4$PO$_4$6H$_2$O), or cystine ($\text{SCH}_2\text{CHNH}_2\text{COOH}$)$_2$. The organic matrix appears to form from proteins that adsorb to the growing crystals.\textsuperscript{8,9} It takes on the structure of a matted sheet between crystals in COM and COD stones.\textsuperscript{10}

COM and uric acid stones have a concentric laminated structure in addition to the crystal arrangement. They tend to break up by separation along the layers.\textsuperscript{5} COM stones also tend to have well-defined radial striations associated with the crystal orientations within the stone.\textsuperscript{9} COM stones are often highly disorganized in the stone center. COD stones show much less structure, unless there is significant COM present also.\textsuperscript{10} Struvite is always present in stones in conjunction with calcium apatite; the stones are coarse grained but quite disorganized.\textsuperscript{7} Struvite stones break differently from COM and uric-acid stones; they tend to fracture irregularly along crystal surfaces.\textsuperscript{11} Cystine stones may have a finer-grained internal structure without laminations or radial striations, or they may be coarse grained. They are either very difficult or very easy to break up by SWL.\textsuperscript{12}

Clearly there is a wide variation in the microstructure of heterogeneous kidney stones. Presently, there appears to be no adequate theory that ascertains the influence of microscale heterogeneity on the stress distributions under loading of kidney stones. The purpose of the present work is to develop a theoretical framework for formulating these problems. What is required is a tractable way of handling the solid mechanics that accounts for the presence of small-scale structure. For simplicity, this first effort considers stones that have a microstructure that can be characterized as random and isotropic, and linearly elastic in both phases. Therefore the present analysis might be expected to apply more directly to cystine and struvite stones, perhaps with some generosity to COD stones, and likely not particularly well to uric-acid and COM stones. The reason is that the latter compositions lead to large-scale structure and anisotropy that violates the simple assumptions that will be made in this first theory.
TABLE I. Material properties used in the present work, estimated from experimental data from ref. 13 as described in the text. The matrix has bulk and shear moduli of $\kappa_1$ and $\mu_1$. The harder crystalline inclusions have bulk and shear moduli of $\kappa_2$ and $\mu_2$. The volume fraction of the matrix $\nu_2$ is taken from ref. 13.

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_1$(GPa)</td>
<td>0.160</td>
</tr>
<tr>
<td>$\mu_1$(GPa)</td>
<td>$\kappa_1^3$</td>
</tr>
<tr>
<td>$\kappa_2$(GPa)</td>
<td>14.973</td>
</tr>
<tr>
<td>$\mu_2$(GPa)</td>
<td>$\kappa_1^3$</td>
</tr>
<tr>
<td>$\nu_2$</td>
<td>0.90</td>
</tr>
</tbody>
</table>

then the product of two second-order tensors $\mathbf{A} \cdot \mathbf{B}$ is defined by the matrix product $[\mathbf{A}] [\mathbf{B}]$, with components of $A_{ij}B_{jk} = C_{ijk}$, where repeated indices imply summation. The scalar product of two tensors or matrices is $\mathbf{A} \cdot \mathbf{B} = A_{ij}B_{ij} = \text{tr} ([\mathbf{A}]^T [\mathbf{B}])$, where $\text{tr} (\cdot)$ is the trace operator. Finally, the divergence of a vector field $\mathbf{u}$ is defined by $\nabla \cdot \mathbf{u} = u_i,i$—with a comma in the subscript denoting differentiation with respect to the $i$th coordinate; for a second-order tensor field $\mathbf{A}$, $\nabla \cdot \mathbf{A}$ describes a contraction to a vector with the components $A_{ki,i}$. With this preamble out of the way, a description of the effective material properties of the heterogeneous body is now given.

**Micro–Macro Properties**

A theory for the mechanics of microheterogeneous kidney stone will require knowledge of the overall effective mechanical properties of microheterogeneous materials, comprised of particles suspended in a binding matrix (Figure 1). Results from the well-developed theory of heterogeneous solids are used in order to determine overall properties. The theory is framed in terms of linear elastic fracture mechanics (LEFM). The basic assumption is that inelastic deformations occur in a region that is small compared to the size of developing cracks. When this is not so, as might be the case if there is significant plastic deformation over large areas of the binding matrix, one should use instead elastic plastic fracture mechanics (EPFM). If a rate-dependent theory is desired, one can make use of viscoelastic material models for the binding matrix or particulate phases. This comes at a significant increase in complexity with many more unknown parameters, and results in models that can only be numerically integrated.

$\mathbf{E}$ denotes the effective mechanical response (or stiffness) —a fourth-order elasticity tensor— described via the relation between average stress and strain fields: $\langle \sigma_{ij} \rangle = \mathbf{E} : \langle \varepsilon_{ij} \rangle$; in terms of components this is $\langle \sigma_{ij} \rangle_{\Omega} = E_{ijkl} \langle \varepsilon_{kl} \rangle_{\Omega}$. Here

$$\langle \cdot \rangle_{\Omega} \overset{\text{def}}{=} \frac{1}{|\Omega|} \int_{\Omega} \cdot d\Omega,$$

and $\sigma$ and $\varepsilon$ are the stress and strain tensor fields within a statistically representative volume element (RVE) of volume $|\Omega|$. The RVE will be discussed shortly. The domain of the matrix phase is $\Omega_1$, and the domain of the particle phase is $\Omega_2$. Of course, at the microscale, the mechanical properties of microheterogeneous materials are characterized by a spatially variable elasticity tensor $\mathbf{E}$. The effective response $\mathbf{E}^*$ is assumed isotropic, which obtains when the particles are randomly distributed and randomly oriented. An isotropic body has material properties that are the same in every direction at a point in the body; that is, the properties are not a function of orientation at a point in the body. It can be shown that when the body is isotropic, there are but two free constants in $\mathbf{E}^*$, which may be written compactly

$$\mathbf{E}^* = \begin{bmatrix} \langle \sigma_{11} \rangle_{\Omega} \\ \langle \sigma_{22} \rangle_{\Omega} \\ \langle \sigma_{33} \rangle_{\Omega} \\ 2\langle \sigma_{12} \rangle_{\Omega} \\ \langle \sigma_{23} \rangle_{\Omega} \\ \langle \sigma_{31} \rangle_{\Omega} \end{bmatrix} = \begin{bmatrix} \kappa^* + \frac{4}{3} \mu^* \\ \kappa^* + \frac{4}{3} \mu^* \\ \kappa^* + \frac{4}{3} \mu^* \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \kappa^* - \frac{2}{3} \mu^* \\ \kappa^* - \frac{2}{3} \mu^* \\ \kappa^* - \frac{2}{3} \mu^* \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Here the effective bulk and shear moduli are given by

$$3 \kappa^* = \frac{\langle \sigma' \rangle_{\Omega}}{\langle \varepsilon' \rangle_{\Omega}} \bigg| \frac{\langle \sigma' \rangle_{\Omega}}{\langle \varepsilon' \rangle_{\Omega}} \bigg|,$$

and

$$2 \mu^* = \frac{\langle \sigma' \rangle_{\Omega}/\langle \sigma' \rangle_{\Omega} \langle \varepsilon' \rangle_{\Omega}/\langle \varepsilon' \rangle_{\Omega} \rangle_{\Omega}},$$

where $\varepsilon' = \varepsilon - \frac{\text{tr} \varepsilon}{3} \mathbf{I}$ is the deviatoric strain, $\sigma'$ is the deviatoric stress, and $\mathbf{I}$ is the second-order identity tensor. In this case one may write
The eigenvalues of an isotropic effective elasticity tensor are $(\kappa^*, \mu^*, 2\mu^*, \mu^*, \mu^* \mu^*, \mu^*)$. Therefore, it is necessary to have $\kappa^* > 0$ and $\mu^* > 0$ to retain positive definiteness of $\mathbf{E}^*$, as is required by thermodynamic arguments. An extensive review of the analysis of random heterogeneous media can be found in the work of Torquato.\textsuperscript{16–20}

It is clear that for concepts involving the relation between averages to be useful, they must be computed over a sample containing a statistically representative amount of material of volume $|\Omega|$. This size requirement can be made precise, as follows. A commonly accepted macro/micro criterion used in effective property calculations is Hill’s condition:\textsuperscript{21} $\langle \sigma : \varepsilon \rangle_\Omega : \langle \varepsilon \rangle_\Omega \leq \kappa^*, \mu^* \leq \kappa_2, \mu_2$, This condition dictates the size requirements on the RVE. The classical argument is as follows. For any perfectly bonded heterogeneous body in the absence of body forces, two physically important loading states satisfy Hill’s condition. They are (a) pure linear displacements of the form $\mathbf{u} \mid_{\Omega} = \varepsilon \cdot \mathbf{x}$ $\Rightarrow \langle \varepsilon \rangle_\Omega = \varepsilon$, and (b) pure tractions in the form $\mathbf{t} \mid_{\Omega} = \mathbf{L} \cdot \mathbf{n}$ $\Rightarrow \langle \sigma \rangle_\Omega = \mathbf{L}$, where $\varepsilon$ and $\mathbf{L}$ are constant strain and stress tensors, respectively. Clearly, for Hill’s conditions to be satisfied within a macroscopic body under nonuniform external loading, the sample must be large enough to possess small boundary field fluctuations relative to its size. Therefore, applying (a)- or (b)-type boundary conditions to a large sample is a way of reproducing approximately what may be occurring in a statistically representative microscopic sample of material in a macroscopic body (Figure 1).

Struvite crystals tend to be up to roughly 20–50 $\mu$m in size; in cystine stones, the crystals are smaller —roughly up to 10 $\mu$m in size\textsuperscript{7} in smooth cystine stones.\textsuperscript{12} It has been estimated that, for the class of microstructures considered, the RVE should be such that it contains between 100 and 1000 particles. For example, see Lemaitre and Chaboche,\textsuperscript{22} Huet,\textsuperscript{23} Zohdi\textsuperscript{24} or Zohdi and Wrighers.\textsuperscript{25} Thus, roughly a (cubical) RVE should contain 5 to 10 particles on a side.

This may be compared to the length scale of a shock. If the rise time of a shock in water is taken to be 10 ns, then the thickness of the shock is about 15 $\mu$m in water, and perhaps 40 $\mu$m in the stone interior. Hence the size of the RVE may be approaching the length scale of the wave front, but is much, much smaller than either the compression or the rarefaction part of the wave. A higher-resolution theory would have to be fully numerical, as in References\textsuperscript{24, 26, and 27}. It is worth emphasizing that in this work, damage is characterized in an overall sense. Local failure mechanisms are inaccessible.

Estimates of the Effective Bulk and Shear Moduli in Terms of Phase Properties

The constitutive tensor $\mathbf{E}^*$ provides the macroscale constitutive properties of a microheterogeneous material because it yields mapping between the average stress and strain measures. Until recently, the direct computation of micromaterial responses was very difficult. Accordingly, classical approaches have sought to approximate or bound effective responses. A widely used set of estimates for the effective properties are the Hashin-Shtrikman bounds\textsuperscript{28, 29} for isotropic materials with isotropic effective responses; for the bulk moduli

$$\kappa^{e+} \equiv \kappa_1 + \frac{v_2}{\kappa_2 - \kappa_1} \left\{ \frac{1}{3} (1 - v_2) \right\} \leq \kappa^* \leq \kappa_2$$

and for the shear moduli

$$\mu^{e+} \equiv \mu_1 + \frac{v_2}{\mu_2 - \mu_1} \left\{ \frac{6(1 - v_2)(\kappa_1 + 2\mu_1)}{5\mu_1(3\kappa_1 + 4\mu_1)} \right\} \leq \mu^* \leq \mu_2$$

where $\kappa_2 \geq \kappa_1$ are the bulk moduli and $\mu_2 \geq \mu_1$ are the shear moduli of the particle and binder phases, respectively, and $v_2$ is the volume fraction of particles; of course the volume fraction of the matrix is $\nu_1$, where $\nu_1 + \nu_2 = 1$. Such bounds are the tightest known (perhaps tightest possible) on isotropic effective responses, with isotropic two-phase microstructures, where only the volume fractions and phase contrasts of the constituents are known. Note that no further geometric information, such as the number and nature of asperities, etc. contributes to these bounds.

A straightforward estimate of the effective properties is to take a convex combination of the bounds; for example,

$$\kappa^* = \theta \kappa^{e+} + (1 - \theta) \kappa^{e-} \quad \text{and} \quad \mu^* = \theta \mu^{e+} + (1 - \theta) \mu^{e-}.$$  

where $0 \leq \theta \leq 1$. In the present work, because hard particles are surrounded by a soft matrix, the properties of the matrix control the effective properties, despite the fact that it is of much smaller volume fraction. Here $\theta = 0.25$. Detailed experimental and analytical work on how to estimate $\theta$ can be found in Zohdi et al.\textsuperscript{30} In short, for microstructures comprised of hard particles surrounded by a continuous soft matrix, which produces an overall stiffness that is significantly smaller than the reverse, a hard matrix encasing soft particles, it is well known that the Hashin-Shtrikman lower bound is quite accurate;\textsuperscript{31} this motivates the choice of $\theta = 0.25$. Note that $\theta$ is a property of the material —measurable or calculable— and not a free parameter. Because it is associated with
the material, all of the material properties are computed with the same $\theta$.

**Load Sharing Between the Two Phases**

The load carried by each phase in the microstructure is characterized via stress concentration tensors, which are now discussed. These provide a measure of the deviation away from the mean fields throughout the material. One can decompose averages of an arbitrary quantity over $\Omega$ into averages over the each of the phases in the following manner:

$$
\langle A \rangle_\Omega = \frac{1}{|\Omega|} \left( \int_{\Omega_1} A \, d\Omega + \int_{\Omega_2} A \, d\Omega \right) = u_1 \langle A \rangle_{\Omega_1} + u_2 \langle A \rangle_{\Omega_2},
$$

(13)

This decomposition leads to

$$
\langle \sigma \rangle_{\Omega_1} = u_1 \langle \sigma \rangle_{\Omega_1} + u_2 \langle \sigma \rangle_{\Omega_2} = u_1 \langle \sigma \rangle_{\Omega_1} + u_2 \langle \sigma \rangle_{\Omega_2} + u_1 \langle \varepsilon \rangle_{\Omega_1} + u_2 \langle \varepsilon \rangle_{\Omega_2}.
$$

Substitution of the identity $\langle \varepsilon \rangle_{\Omega_1} = \langle \varepsilon \rangle_{\Omega} - u_2 \langle \varepsilon \rangle_{\Omega_2}$ leads to

$$
u_1 \langle \varepsilon \rangle_{\Omega_1} + u_2 \langle \varepsilon \rangle_{\Omega_2} = \langle \varepsilon \rangle_{\Omega} - u_2 \langle \varepsilon \rangle_{\Omega_2} + u_2 \langle \varepsilon \rangle_{\Omega_2} = \langle \varepsilon \rangle_{\Omega_1}.
$$

The entire expression may then be rewritten as one involving a premultiplier of $\langle \varepsilon \rangle_{\Omega}$:

$$
\langle \sigma \rangle_{\Omega_1} = \langle \varepsilon \rangle_{\Omega_1} - v_2 \langle \varepsilon \rangle_{\Omega_2} + v_2 \langle \varepsilon \rangle_{\Omega_2} + \langle \varepsilon \rangle_{\Omega_2} - \langle \varepsilon \rangle_{\Omega_1} = ((1 + v_2(\langle \varepsilon \rangle_{\Omega} - \langle \varepsilon \rangle_{\Omega_1})))C;\langle \varepsilon \rangle_{\Omega_1},
$$

where $C; \langle \varepsilon \rangle_{\Omega_1}$, with

$$
C = \frac{1}{v_2} (\langle \varepsilon \rangle_{\Omega_2} - \langle \varepsilon \rangle_{\Omega_1})^{-1} (\langle \varepsilon \rangle_{\Omega} - \langle \varepsilon \rangle_{\Omega_1}).
$$

The strain concentration tensor $C$ relates the average strain over the particle phase (2) to the average strain over all phases. Similarly, for the variation in the stress, $C; \langle \sigma \rangle_{\Omega_1} = \langle \sigma \rangle_{\Omega_1} - \langle \sigma \rangle_{\Omega_2}$, which reduces to

$$
\langle \sigma \rangle_{\Omega_1} = \langle \sigma \rangle_{\Omega_2} + \langle \sigma \rangle_{\Omega} - \langle \sigma \rangle_{\Omega_1}.
$$

(14)

$\bar{C}$ is known as the stress concentration tensor; it relates the average stress in the particle phase to that in the whole RVE. Note that once either $\bar{C}$ or $E^*$ are known, the other can be determined.

In the case of isotropy

$$
\tilde{C}_\kappa = \frac{\kappa_2 - \kappa_1}{\kappa_2 - \kappa_1} \text{ and } \tilde{C}_\mu = \frac{\mu_2 - \mu_1}{\mu_2 - \mu_1},
$$

(15)

and where $\bar{C}_\kappa (\langle \sigma \rangle_{\Omega_1}) = \langle \sigma \rangle_{\Omega_2}$. Clearly, the microstress fields are minimally distorted when $\bar{C}_k = \bar{C}_\mu = 1$; there are no stress concentrations in a homogenous material, for homogeneous loading. For the matrix,

$$
\langle \sigma \rangle_{\Omega_1} = \frac{\langle \sigma \rangle_{\Omega_1} - u_2 \bar{C}_\kappa (\langle \sigma \rangle_{\Omega_1})}{u_1} = \frac{(1 - u_2 \bar{C}_\kappa (\langle \sigma \rangle_{\Omega_1})}{u_1} = C;\langle \sigma \rangle_{\Omega_1}.
$$

(16)

Therefore, in the case of isotropy,

$$
C_\kappa = \frac{1}{u_1} (1 - u_2 \bar{C}_\kappa) \text{ and } C_\mu = \frac{1}{u_1} (1 - u_2 \bar{C}_\mu).
$$

(17)

The utility of such relations is that they allow one to determine what the load sharing is for each phase. Typically, for kidney stones, the failure of the matrix that binds the particulates together is of interest. However, the theory developed here can be applied to estimate failure of the matrix or of the particulates.

**FATIGUE-LIFE ESTIMATES FOR THE DESTRUCTION OF KIDNEY STONES**

The classical Basquin relation is employed for fatigue-life estimation, which is as follows at a material point,$^{32}$

$$
\| \sigma \| = \| \langle \sigma \rangle - \| \sigma_m \| \| N_f \| = \frac{1}{2} \frac{\| \sigma_m \|^2}{\| \sigma_m \|^2} = \frac{1}{2} \frac{\| \sigma_m \|^2}{\| \sigma_m \|^2},
$$

(18)

where $\| \sigma \| = \sqrt{\langle \sigma \rangle \cdot \sigma}$. Here the norm of the mean stress is $\| \sigma_m \| = \| \sigma_{max} + \sigma_{min} \| / 2$, and the norm of the fluctuating stress is $\| \sigma_f \| = \| \sigma_{max} - \sigma_{min} \| / 2$. Static failure stress is denoted by $\sigma_f$ and, typically for most known materials, $-0.12 \leq b \leq -0.05$. Classical relations of this type are discussed in Suresh.$^{33}$ Classical approaches are followed here, and $\sigma_f$ is set to be the minimum of the absolute values of the failure stresses in tension or compression, when these differ.

This is extended to overall failure of each of the phases by using volumetric averages over each phase. For the second
failure criterion for the matrix satisfy is considered. With this loading, all states which satisfy the
where, due to the overall hydrostatic loading, the overall failure depends on only the bulk modulus concentration
of the previous expressions are substituted into (22), the relationship between \( N_f \), \( A \), and \( B \) becomes

\[
N_f = \frac{1}{2} \left( \frac{\| \langle \sigma \rangle_{\Omega} \|}{\| \langle \sigma_m \rangle_{\Omega} \|} \right)^{\frac{1}{b_1}} = \frac{1}{2} \left( \frac{\| \overline{C} : \langle \sigma \rangle_{\Omega} \|}{\| \overline{C} : \langle \sigma_m \rangle_{\Omega} \|} \right)^{\frac{1}{b_1}},
\]

\[ (22) \]

The rates of increase and decrease of the fatigue lives are
Consistent with the use of effective (volumetrically averaged) properties, a body is considered with an idealized
loading on its boundary of the form \( t_{\Omega} = L \cdot n \), where \( t \) is the
the surface traction, \( L \) is a second-order (load) tensor, and \( n \) is the outward unit normal on the boundary \( \partial \Omega \). Consistent
with classical fatigue theory, inertial effects are neglected. The identity \( \nabla \cdot (\sigma \otimes x) = (\nabla \cdot \sigma) \otimes x + \sigma \cdot \nabla x = \sigma \) is used
where \( x \) is the position vector of an arbitrary point within the
body, and \( \otimes \) is the dyadic product. This is substituted into the
definition of the average stress to obtain (using Gauss’s
divergence theorem)

\[
\langle \sigma \rangle_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} \nabla \cdot (\sigma \otimes x) \, d\Omega = \frac{1}{|\Omega|} \int_{\partial \Omega} (\sigma \otimes x) \cdot n \, da
\]

\[ (21) \]

\[
= \frac{1}{|\Omega|} \int_{\partial \Omega} (L \otimes x) \cdot n \, da = L
\]

For simplicity, pressure-like loading on the surface of the
RVE in the form

\[
t_{\partial \Omega} = \begin{bmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.
\]

\[ (21) \]

is considered. With this loading, all states which satisfy
the failure criterion for the matrix satisfy

\[
N_f = \frac{1}{2} \left( \frac{\| \langle \sigma \rangle_{\Omega} \|}{\| \langle \sigma_m \rangle_{\Omega} \|} \right)^{\frac{1}{b_1}} = \frac{1}{2} \left( \frac{\| \overline{C} : \langle \sigma \rangle_{\Omega} \|}{\| \overline{C} : \langle \sigma_m \rangle_{\Omega} \|} \right)^{\frac{1}{b_1}},
\]

\[ (20) \]

The compact notation is employed: \( \langle \sigma \rangle_{\Omega} = \frac{1}{n} \langle \sigma \rangle_{\Omega} \cdot \frac{1}{n} \langle \sigma \rangle_{\Omega} \),
\( \overline{C} : \langle \sigma \rangle_{\Omega} = \overline{C} : \langle \sigma \rangle_{\Omega} \cdot \frac{1}{n} \langle \sigma \rangle_{\Omega} \).

Of course, the RVE experiences both dilatational and deviatoric
loading. Stress concentrations for both dilatational and deviatoric loading have been developed. With a suitable
mesoscopic model that includes full coupling between the fluid
and stone one could be definite about what loading to prescribe on the RVE. Such a model does not yet exist. This
first examination of the influences of stress concentrations
associated with microscale heterogeneity will focus on meso-
scopic loading of the following form: \( \langle \sigma \rangle_{\Omega} = A \langle \sigma \rangle_{\Omega} \) (tension, \( A > 0 \)) and \( \langle \sigma \rangle_{\Omega} = B \langle \sigma \rangle_{\Omega} \) (compression, \( B < 0 \)), with a normalization of

\[
\langle \sigma \rangle_{\Omega} = \sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

\[ (23) \]

Here \( \sigma_f \) is the failure stress of the binding matrix. Note that
here the focus is on Mode I (opening mode) of microcrack
growth due to fatigue, hence the form of the loading considered.
Therefore, the amplitude of the alternating stress is

\[
\langle \sigma \rangle_{\Omega} = \frac{A - B}{2} \langle \sigma \rangle_{\Omega},
\]

\[ (24) \]

and the mean is

\[
\langle \sigma_m \rangle_{\Omega} = \frac{A + B}{2} \langle \sigma \rangle_{\Omega},
\]

\[ (25) \]

Thus, if the previous expressions are substituted into (22), the
relationship between \( N_f \), \( A \), and \( B \) becomes

\[
N_f = \frac{1}{2} \left( \frac{\| \langle \sigma \rangle_{\Omega} \|}{\| \langle \sigma_m \rangle_{\Omega} \|} \right)^{\frac{1}{b_1}} = \frac{1}{2} \left( \frac{\| \overline{C} : \langle \sigma \rangle_{\Omega} \|}{\| \overline{C} : \langle \sigma_m \rangle_{\Omega} \|} \right)^{\frac{1}{b_1}},
\]

\[ (26) \]

where, due to the overall hydrostatic loading, the overall
failure depends on only the bulk modulus concentration
factor, and on the maximum and minimum averaged dimension-
less stresses. Note that in an overall, volumetrically averaged
sense, overall deviatoric stresses vanish, and local deviatoric
failures cannot be resolved, due to the type of loading con-
sidered, that is, Mode I opening of a microscale crack.

The number of cycles to failure versus maximum (dimen-
sionless) tension \( A > 0 \) and compression \( B < 0 \) per cycle is
plotted in Figure 2, using material properties found in Table I.
The fatigue exponent \( b_1 \) was chosen to be the average of the
ranges cited earlier \(-0.085 = -(0.12 + 0.05)/2\). It is em-
phazisized that \( b_1 \) is a material property —measurable or cal-
culable—and not a free parameter, in principle. These results
for the number of cycles to failure fall within the ranges
typically needed for the destruction of kidney stones (\( 10^2 -
10^4 \)). With the parameters determined as indicated, it was
found that $C_A/H_{9260}/H_{11005}/0.273$. The value of $C_{A/H_{9262}}$ is irrelevant due to the overall purely hydrostatic stress state. This implies that the external loading, characterized by $A$ and $B$, will induce significantly smaller loads on the softer binding matrix than on the harder inclusions. Therefore, the peak stresses made dimensionless by the matrix failure stress, $A$ and $B$, must be significantly greater than unity to induce failure of the matrix. If the reasoning of Cohen and Whitfield$^{34}$ is followed, then the magnitudes may be reckoned in this way. From Figure 2, one can see that about 350 cycles are required to fragment a stone if $A = -B \approx 2.3$. If the matrix failure stress in tension is 5 bar, then $A = -B \approx 2.3$ corresponds to maximum and minimum stresses of roughly 10 bar. Owing to energy losses at the stone–fluid interface, this is associated with a liquid pressure of at least three times, or 30 bar. Finally, attenuation of the shock wave as it travels through (say 10 cm of) tissue would require a pulse of about 10 times that amount, or 300 MPa. This is within the range of lithotripters, although likely it is true that $A \neq -B$.

Of course, the nature of the cyclic loading at any one point in the stone is best derived from a mesoscopic model that accounts for coupling to the surrounding medium, and reflection and interaction of stress waves. Such a model for pressure waves in a homogeneous material was recently developed by Cleveland and Tello.$^{35}$ As one would expect, if the mean stress is increased, the alternating stress can be decreased to induce fatigue failure. Furthermore, if it is possible to increase the number of cycles, one can decrease the intensity of the mean and alternating stresses.

Some statements can be made about the influence of $b_1$ and $\theta$ on the fatigue surface shown in Figure 2. Alteration of $b_1$ changes the slope of the surface; whereas changes in $\theta$ shift it up or down dramatically; see Figure 3. Hence these two parameters, once measured, are essential ingredients in a sound theory.

**DISCUSSION AND CONCLUSIONS**

In the course of this work a number of approximations and estimates were made. The goal was to develop a theoretical framework capable of handling the mechanics of microheterogeneous kidney stones. To that end, an isotropic, random microstructure with two phases—particulate and binder—was considered. Stress concentration functions were developed to determine load sharing between the particle phase and the binding matrix phase. To illustrate application of the theory, fatigue-life estimates were developed for each phase, as a function of the volume fraction and of the mechanical properties of the constituents, as well as the mean and fluctuating loading. The failure of the binding matrix could then be predicted explicitly.

Measurements of laminated uric-acid stones$^{13}$ were used to estimate properties of matrix and binder, although this analysis applies more directly to struvite and cystine stones. Needless to say, the theory could be improved significantly by property measurements in a variety of stones. The analysis has also made use of a number of approximations. There remain material properties that could be measured in an experiment. These include the fatigue parameters in each phase $b_i$ and the weighting $\theta$ in the mean property estimates. Some methods for determination of these parameters from experiments have been described.$^{30}$ Of course, it would also be useful to measure directly the bulk and shear moduli of the binding matrix and particulate phases; then one would not have to resort to determination of these quantities by the methods used earlier. Finally, it would be worthwhile to measure the failure stress for both matrix and particulate phases, in order to yield quantitative estimates for fatigue.
failure. The approach presented here can serve as a precursor to more elaborate large-scale computational simulations, such as those found in Zohdi.34, 26, 27 The assumption of isotropy could be relaxed, as estimates for certain anisotropic responses can be found in Hashin.31 This would allow one to treat a greater variety of stones.

With a deeper understanding of the differences in the destruction of kidney stones of different types, one could possibly tailor the number and intensity of cycles of loading based on the type of stones. In vivo, one could imagine that the morphology could be determined by inverse methods, such as x-ray tomography. Such information could be used to obtain more precise estimates of the material properties, for use in a model that would estimate the number of cycles to failure. One can target chemical alteration of the stone with a theory of the kind presented here. If, for example, the particulate phase is weakened, particulates will carry a smaller fraction of the load, hence weakening the stone.

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